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Basic Algorithms for Digital Image Analysis: a course

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Lecture 12: Binary image processing II

Morphological processing

- Basic operations: erosion, dilation
- Properties of erosion and dilation
- Opening and closing
- Other morphological operations:
  - Hit-Miss
  - Boundary
  - Medial axis
  - Thinning and thickening
  - Pruning

Basic notions of morphological processing

Morphology: Study of structure and form of animals, plants, or words, phrases.

- Express structures and forms in terms of **structuring elements**.

Mathematical morphology: Study of structure and form of images or other spatial structures by comparing them to a sliding **structuring element**.

- Hit objects with structuring element, transform them to more revealing shapes.

Most morphological operations can be defined in terms of two basic operations, **erosion** and **dilation**

B - **structuring element** with **origin** c. B is a discrete set of points of specific configuration aimed at particular operations.

B_x - **translation** of B so as to have the origin c in point x of **digital image** X.

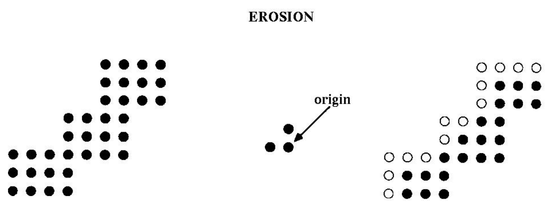
- The output of a morphological operation is assigned to x.

Erosion

Erosion of X by B is the set of all points x such that B_x is included in X:

$$X \ominus B = \{x : B_x \subset X\}$$

In other words, nonzero output is assigned to those positions of the origin in which **all points** of the structuring element coincide with nonzero (object) points of the image.



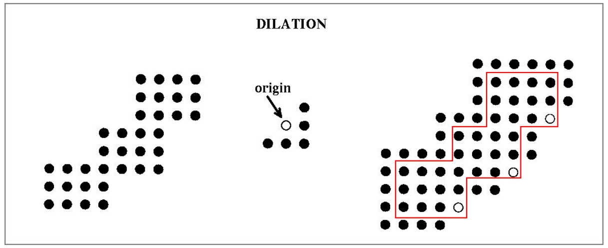
Example of erosion. The empty circles are the deleted points.

Dilation

Dilation of X by B is the set of all points x such that B_x hits X, that is, they have a nonempty intersection:

$$X \oplus B = \{x : B_x \cap X \neq \emptyset\}$$

In other words, nonzero output is assigned to those positions of the origin in which **at least one point** of the structuring element coincides with a nonzero (object) point of the image.



Example of dilation. Empty circles in the dilated image are the deleted points. The contour shows the original object.

Properties of erosion and dilation

1. **Translation invariance**: Shift of object leads to the same shift in result.

2. Dilation is **commutative**: $X \oplus B = B \oplus X$, erosion is not.
   - But $(X \ominus A) \ominus B = (X \ominus B) \ominus A$

3. Erosion and dilation are **not inverses of each other**.

4. **Distributivity in structuring element**:

$$X \oplus (B \cup B') = (X \oplus B) \cup (X \oplus B')$$
$$X \ominus (B \cup B') = (X \ominus B) \cap (X \ominus B')$$

   A structuring element can be represented, or approximated, by **union** of two or more simple elements.

   - For instance, disc can be approximated by union of 1 square and 2 rectangles to **speed up erosion**: erosion by rectangle is implemented as running operation.

5. **Distributivity in image**, or local knowledge:

$$(X \cap Y) \ominus B = (X \ominus B) \cap (Y \ominus B)$$
$$(X \cup Y) \oplus B = (X \oplus B) \cup (Y \oplus B)$$

6. **Iteration**:

$$(X \ominus B) \ominus B' = X \ominus (B \oplus B')$$
$$(X \oplus B) \oplus B' = X \oplus (B \oplus B') \quad \textbf{(associativity)}$$

   When a structuring element can be represented as **dilation** of two or more smaller elements, erosion and dilation can be performed faster.

   - For instance, erosion by 9×9 square can be implemented as two subsequent erosions by 5×5 square, since the 9×9 square is dilation of two 5×5 squares.

7. **Increasing**:

   If $X \subset X'$, then

$$X \ominus B \subset X' \ominus B \quad \forall B$$
$$X \oplus B \subset X' \oplus B \quad \forall B$$

   If $B \subset B'$, then

$$X \ominus B' \subset X \ominus B \quad \forall B$$
$$X \oplus B \subset X \oplus B' \quad \forall B$$

8. **Duality** with respect to the complement operation:

$$\underline{X} \oplus B^{\sim} = \underline{X \ominus B}$$

   where $\underline{X}$ is complement of X, $B^{\sim}$ reflection of B with respect to the origin.

Opening and closing

- **Opening** of set X by structuring element B:

$$X_B = (X \ominus B) \oplus B$$

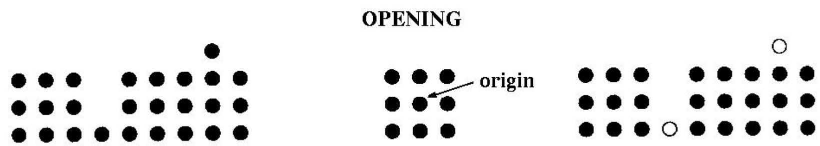
  - Smooths contours, suppresses small islands and sharp caps
  - Can be used for object size distribution study

- **Closing** of set X by structuring element B:

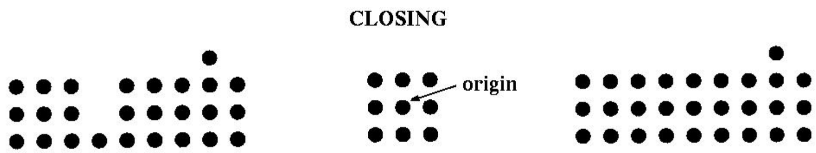
$$X^B = (X \oplus B) \ominus B$$

  - Fills up narrow channels and thin lakes
  - Can be used for inter-object distance study

- Opening and closing are **idempotent** operations and are **duals of each other**:
  - $(X_B)_B = X_B$ and $\left(X^B\right)^B = X^B$
  - $\underline{(X_B)} = (\underline{X})^B$ and $\underline{\left(X^B\right)} = (\underline{X})_B$



Example of opening.



Example of closing.

Hit-Miss: Matching binary patterns

Hit-Miss: Search for a **match** or a specific configuration.

In this operation, one has to specify both object **and** background points of a structuring element.

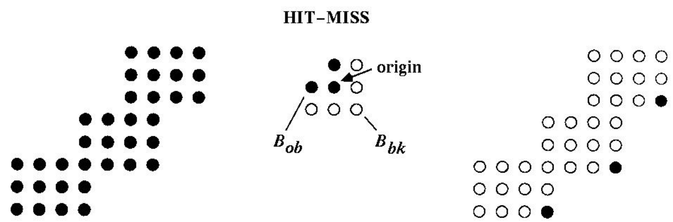
- $B_{ob} \subset B$: object part of structuring element

- $B_{bk} \subset B$: background part of structuring element

The **hit-miss** operator outputs an object pixel in the positions where B_{ob} matches the object pixels **and** B_{bk} matches the background pixels:

$$X \otimes B = \{x : B_{ob} \in X \text{ and } B_{bk} \in \underline{X}\} = (X \ominus B_{ob}) \cap (\underline{X} \ominus B_{bk})$$

Comment: In the definition, B_{bk} is treated as consisting of **object** type pixels that have to match the **complement** $\underline{X}$.



Finding corners with hit-miss.

Comments:

- Hit-miss corresponds to pattern matching in greyscale images.

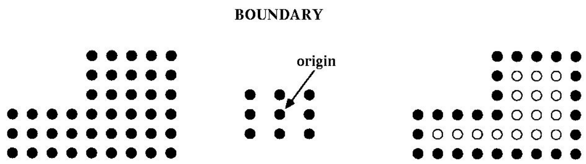
- Other definitions of hit-miss also exist; they are **equivalent** to our definition.

- Symbol $\odot$ is also used to denote the hit-miss.

Boundary

Boundary of set X: $\partial X = X \setminus X \ominus G$

- The interior points obtained by the **erosion** are subtracted from the original image. The difference is formed by the boundary points.
- The structuring element is $G = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$
  - G is a simple approximation of small **digital disc** frequently used in morphological processing.



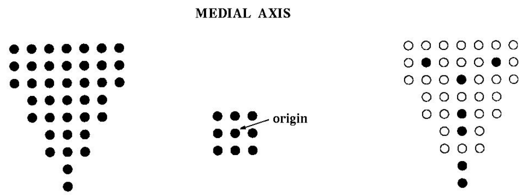
Example of boundary extraction.

Medial axis

Medial axis $S(X)$ of set X is obtained as union of its parts $s_n(X)$ using the structuring element G introduced above:

$$S(X) = \bigcup_{n=0}^{n_{max}} [(X \ominus nG) \setminus (X \ominus nG)_G] \doteq \bigcup_{n=0}^{n_{max}} s_n(X)$$

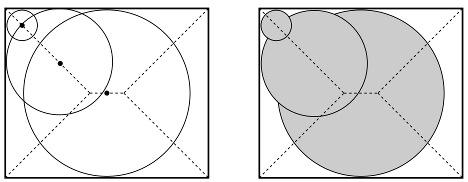
- Notations:
  - n_{max}: maximum size after which X erodes down to empty set
  - $(X \ominus nG)$: nth iteration $(X \ominus G) \ominus G \ominus \cdots \ominus G$
  - $(X \ominus nG)_G$: opening of $(X \ominus nG)$ by G



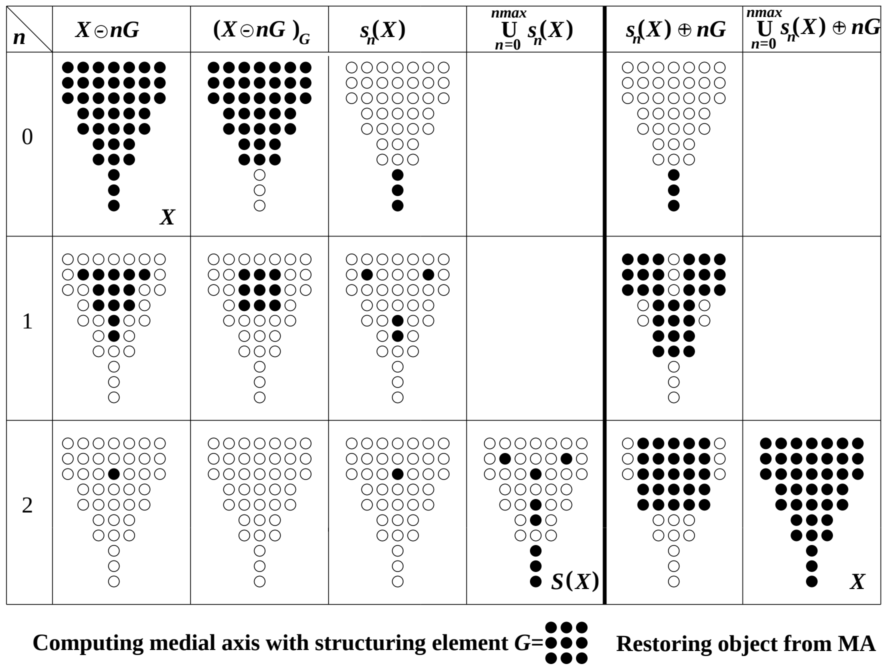
Restoring set X from its medial axis:

$$X = \bigcup_{n=0}^{n_{max}} [s_n(X) \oplus nG]$$

- MA is detected as set of centers of maximal discs that are contained in X and touch the boundary of X in two or more locations.
- To restore the object from its MA, we take union of circular neighborhoods centered on MA points and having radii equal to associated contour distances.



Morphological MAT. Left: Medial axis as locus of inscribed circles (erosions). Right: Regeneration of shape from MA using dilations by discs.



Thinning and thickening

Thinning of set X is performed by iteratively applying **eight rotated versions** of a structuring element L until no changes occur. It uses **hit-miss** ($\otimes$) and transforms an object to a set of branches, roughly along the medial axis.

$$\begin{aligned} X \bigcirc \{L\} &= ((\ldots((X \bigcirc L^1) \bigcirc L^2)\ldots) \bigcirc L^n) \\ X \bigcirc L^i &= X \setminus (X \otimes L^i) \end{aligned}$$

L^i is L^{i-1} rotated by $45°$, with $L^1 = L \doteq \begin{matrix} 1 & 1 & 1 \\ \mathrm{d} & 1 & \mathrm{d} \\ 0 & 0 & 0 \end{matrix}$, $L^2 = \begin{matrix} \mathbf{d} & 1 & \mathrm{d} \\ 1 & 1 & 0 \\ \mathrm{d} & 0 & 0 \end{matrix}$,. . .,

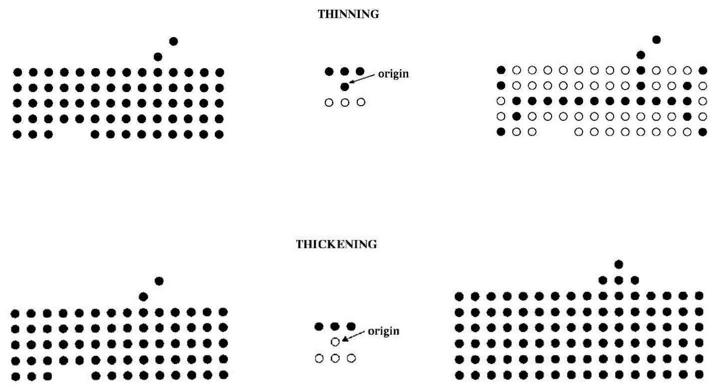
where ‘d’ means any value: ‘don’t care’. (Note **d** in L^2!)

Thickening is the dual of thinning, that is, thinning of background.

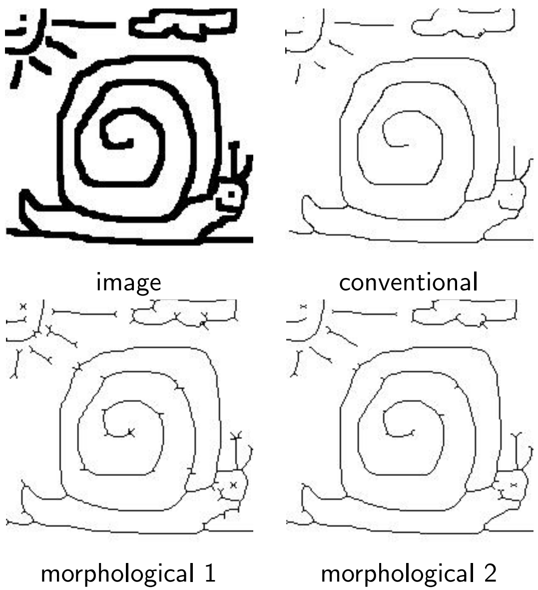
$$X \odot L_b = X \cup (X \otimes L_b)\,,$$

where $L_b = \underline{L} = \begin{matrix} 0 & 0 & 0 \\ \mathrm{d} & 0 & \mathrm{d} \\ 1 & 1 & 1 \end{matrix}$ is rotated in the same way as in thinning.

- A single thickening involves 8 operations $X \odot L_b$.
- Results of thinning and thickening may depend on the order of rotations.



Examples of thinning (top) and thickening (bottom).



Comparison of conventional and morpological thinnings. Morphological 1 is standard rotation of L. In morphological 2, skewed rotations were applied first.

Pruning

Pruning eliminates small parasite branches of the object. It is often applied after morphological MAT or thinning.

$$\begin{aligned} X_1 &= X \bigcirc \{E\} && (1) \\ X_2 &= \bigcup_{j=1}^{8} (X_1 \otimes E^j) && (2) \\ X_3 &= (X_2 \oplus \{G\}) \cap X && (3) \\ X_{pn} &= X_1 \cup X_3 && (4) \end{aligned}$$

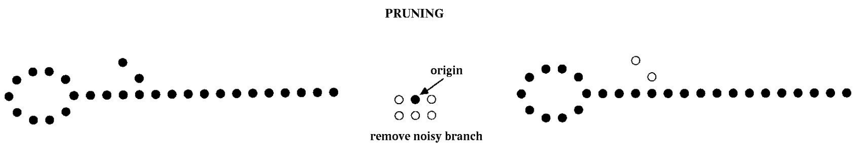
where

$$E = \begin{matrix} \mathrm{d} & \mathrm{d} & \mathrm{d} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}\;, \quad G = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

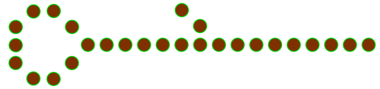
- In (1), eight rotated versions of E are used in cascade. This **recursively removes the brances**. The number of recursions defines the maximum length of branches to be removed.

- In (2), the **end points** of the remaining branches **are detected**.

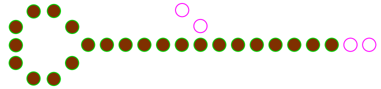
- In (3), the remaining branches are **recursively 'grown back'** to the original size by dilation, to obtain the final pruned object X_{pn} with the short branches suppressed.
  - The dilation is limited to the original set, which is used as a **marker set**. Such operation is called **geodesic dilation.**



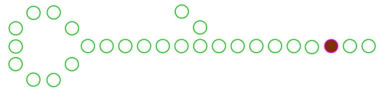
Example of pruning.



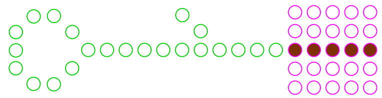
X: original object



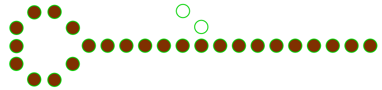
X_1: branches deleted



X_2: endpoint detected



X_3: endpoint grown back



$X_{pn} = X_1 \cup X_3$: pruned object

- $X_1 = X \bigcirc \{E\}$ - applied 2 times, eliminates branches of length 2.

- $X_2 = \bigcup_{j=1}^{8} (X_1 \otimes E^j)$ - detects endpoints of X_1.

- $X_3 = (X_2 \oplus \{G\}) \cap X$ - end of useful branch is grown back to original size.

Summary of morphological processing

- Morphological operations can be naturally **extended to greyscale images**.

- They are usually used for the following purposes:
  - Image pre-processing (noise filtering, shape simplification)
  - Enhancing object structure (skeletonisation, thickening)
  - Segmenting objects from background
  - Quantitative description of objects (area, perimeter, projections, holes)

- Morphological operations are best suitable for processing and statistically describing images containing **many small objects**.

- They are less suitable for precise description of **large**, complex shapes.

- A drawback of morphological processing is its **sensitivity to image orientation**: many operations are rotation-sensitive.