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Basic Algorithms for Digital Image Analysis:

a course

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Lecture 9: Grey-level thresholding

- Principles of grey-level thresholding
- Histogram-based thresholding
- Methods for histogram-based threshold selection
  - Histogram **modality analysis**
  - **Best separation** of classes (Otsu)
  - **Histogram modelling** by Gaussian distributions
- Discussion of grey-level thresholding
  - Examples of thresholding
  - Imroving the histogram for better peak separation
  - Thresholding versus edge detection
  - Limits of thresholding

Principles of grey-level thresholding

Grey-level thresholding is a simple **image segmentation** technique that assumes the following **conditions**:

- Scene model: Scene contains uniformly illuminated, flat surfaces.
- Image model: Image is a set of approximately uniform regions.

Goals of thresholding: Set one or more **thresholds** which split the intensity range into intervals defining **intensity classes**

- Separate objects from background.
- Label objects by classifying pixel intensities into two or more classes.

Definition of N-level thresholding

Set $N-1$ thresholds T_k, $k = 1, \dots, N-1$, $N \geq 2$, so that a pixel $f(x, y)$ is classified into class n if

$$T_{n-1} \leq f(x,y) < T_n, \quad n = 1, \dots, N,$$

where by definition $T_0 \doteq G_{min}$ and $T_N \doteq G_{max} + 1$ are the limits of the intensity range (0 and 256).

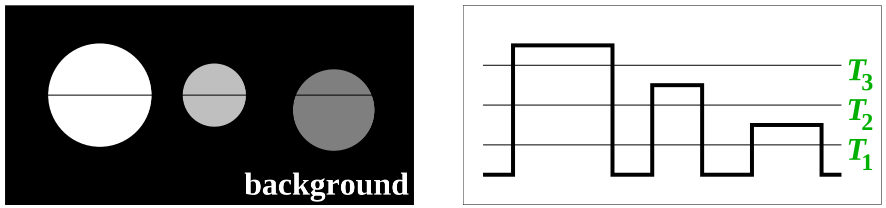


Illustration of 4-level thresholding. By definition, $T_0 = 0$ and $T_4 = 256$. The first level is the background.

original image | 2-level thresholding | 3-level thresholding

Examples of automatic thresholding into 2 and 3 levels.

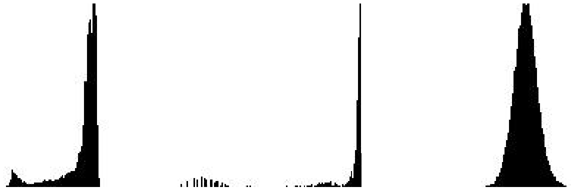
- The case of a single threshold ($N = 2$) is called **bilevel** (binary) thresholding, or **binarisation**.

$\Longrightarrow$ The case considered in this course.

- If $N > 2$, thresholding is called **multilevel**.
  - Sometimes, the case $N = 3$ is called **trilevel** thresholding.

Histogram-based thresholding

- **Bimodal histogram** with distinct modes and valley between modes is **most suitable** for threshold selection. **Minimum of valley** separates the 2 classes.
- If a mode lies at limit of intensity range, modelling the histogram is difficult.
- If modes are not distinct, setting a good threshold is not easy.
- Thresholding a **unimodal histogram** is difficult but still possible.



bimodal | too small, too close | unimodal

Typical histogram shapes for threshold selection. From left to right: bimodal histogram; mode is too small, peak is too close to limit; unimodal histogram.

Examples of good and bad threshold selections for a fingerprint image:

- Different thresholds are acceptable.
- A **too low** threshold tends to split the lines.
- A **too high** threshold tends to merge the lines.

Image | Good 1 | Good 2 | Too low | Too high | Histogram

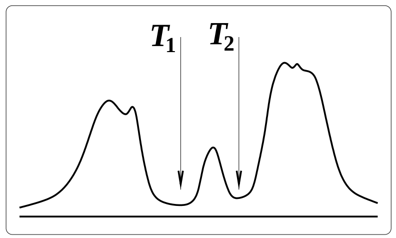
Thresholding a fingerprint image. In the histogram, positions of good (G), too low (L) and too high (H) thresholds are shown.

Histogram modality analysis

Algorithm: Select threshold(s) in valley(s) between peaks.

Parameters:

- Minimum height of peak
- Minimum distance between peaks



Histogram modality analysis: Selecting thresholds in valleys between peaks.

Advantages of modality analysis:

- Natural and easy to understand.
- Multilevel thresholding possible.
- Relatively small populations (classes) can be treated, at least in principle.

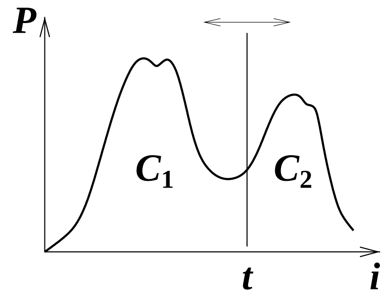
Drawbacks of modality analysis:

- Subjective: What is a peak? A valley?
- Several parameters should be preset that specify these histogram features.
- Many histograms are not multimodal
  - Unimodal histograms
  - Histograms having no clear modes
  - Possible solution: Modify histogram to obtain **distinct modes** (discussed later)

Maximal separation of classes (N.Otsu, 1978)

Basic idea:

- Consider a **candidate threshold** t. t defines two classes of grayvalues.
- Find the optimal threshold $t = t_{opt}$ as the one that maximises a **separation measure** for the two classes.



Obtaining the best possible separation of two classes.

Consider the **normalised histogram** $P(i)$, $i = 0, 1, \dots, M$. It has **mean** μ and **variance** σ^2:

$$\mu = \sum_{i=0}^{M} i \cdot P(i) \qquad \sigma^2 = \sum_{i=0}^{M} (i - \mu)^2 \cdot P(i) \tag{1}$$

A candidate threshold t splits the histogram into 2 classes whose means and variances are

$$\mu_k(t) = \frac{1}{q_k(t)} \sum_{i=a_k}^{b_k} i \cdot P(i) \qquad \sigma_k^2(t) = \frac{1}{q_k(t)} \sum_{i=a_k}^{b_k} [i - \mu_k(t)]^2 \cdot P(i)$$

where $k = 1, 2$, $a_1 = 0$, $b_1 = t$, $a_2 = t + 1$, $b_2 = M$ and

$$q_k(t) = \sum_{i=a_k}^{b_k} P(i) \qquad q_1(t) + q_2(t) = 1$$

Introduce **between-class variance** $\sigma_B^2(t)$ and **within-class variance** $\sigma_W^2(t)$:

$$\begin{aligned} \sigma_B^2(t) &= q_1(t) \cdot [1 - q_1(t)] \cdot [\mu_1(t) - \mu_2(t)]^2 \\ \sigma_W^2(t) &= q_1(t) \cdot \sigma_1^2(t) + q_2(t) \cdot \sigma_2^2(t) \end{aligned} \tag{2}$$

It is easy to show that

$$\mu = q_1(t) \cdot \mu_1(t) + q_2(t) \cdot \mu_2(t) \qquad \sigma^2 = \sigma_W^2(t) + \sigma_B^2(t)$$

Since $\sigma_W^2(t) + \sigma_B^2(t)$ is constant, we have two equivalent options:

- $\sigma_B^2(t)$ is a measure of **class separation** $\Rightarrow$ **Maximise** $\sigma_B^2(t)$
- $\sigma_W^2(t)$ is a measure of **class overlap** $\Rightarrow$ **Minimise** $\sigma_W^2(t)$

We use the **first option**.

To compute $\sigma_B^2(t)$ for any discrete $t > 0$, **recursive formulae** are used:

$$
\begin{aligned}
q_1(t+1) &= q_1(t) + P(t+1) \quad \text{with} \quad q_1(0) = P(1) \\
\mu_1(t+1) &= \frac{q_1(t) \cdot \mu_1(t) + (t+1) \cdot P(t+1)}{q_1(t+1)} \quad \text{with} \quad \mu_1(0) = 0 \\
\mu_2(t+1) &= \frac{\mu - q_1(t+1) \cdot \mu_1(t+1)}{1 - q_1(t+1)}
\end{aligned}
\tag{3}
$$

Algorithm 1: Otsu threshold selection

1. Compute normalised histogram $P(i)$ of image $I(r, c)$.
2. Starting from $t = 0$ and using (3) and (1), recursively compute $q_1(t)$, $\mu_1(t)$ and $\mu_2(t)$ for each $t < G_{max}$.
3. For each $0 < t < G_{max}$, calculate $\sigma_B^2(t)$ by (2).
4. Select threshold as $t_{opt} = \arg\max_t \sigma_B^2(t)$.

Properties of Otsu threshold selection

Advantages:

- General: No specific histogram shape assumed.
- Works well, stable.
- Extension to **multilevel thresholding** possible.
  - For N thresholds and $M+1$ grey levels, optimisation of class separation needs maximum search in a $(M+1)^N$ array

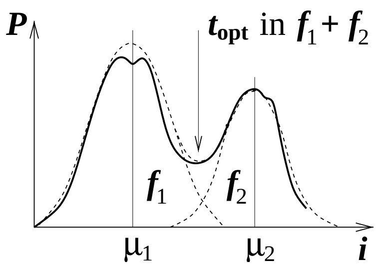
Drawbacks:

- The method assumes that $\sigma_B^2(t)$ is unimodal. This is not always true.
- When optimisation function is flat, false maxima may occur.
- The method tends to artificially **enlarge small classes** to obtain 'better separation': small classes may be merged and missed.

Histogram modelling by Gaussian distributions

Basic idea:

- Assume that the histogram $P(i)$ is a **mixture of Gaussian distributions**
- Approximate $P(i)$ by this model and estimate the parameters of the model
- Find optimal threshold(s) analytically as valley(s) in the model function



Modeling the histogram by a mixture of two Gaussian distributions.

Theoretically correct estimation is possible in the case of **two** Gaussian distributions, that is, for **bilevel** thresholding.

Approximate the histogram $P(i)$ with the model distribution

$$
\begin{aligned}
f(i; \mathbf{p}) &= q_1 \cdot f_1(i; \mathbf{p_1}) + q_2 \cdot f_2(i, \mathbf{p_2}) \\
&= \frac{q_1}{\sqrt{2\pi}\sigma_1} \exp -\frac{1}{2}\left(\frac{i-\mu_1}{\sigma_1}\right)^2 + \frac{q_2}{\sqrt{2\pi}\sigma_2} \exp -\frac{1}{2}\left(\frac{i-\mu_2}{\sigma_2}\right)^2,
\end{aligned}
\tag{4}
$$

where $\mathbf{p} = (q_1, q_2, \mu_1, \mu_2, \sigma_1, \sigma_2)$, $\mathbf{p_1} = (q_1, \mu_1, \sigma_1)$, $\mathbf{p_2} = (q_2, \mu_2, \sigma_2)$ are the parameter sets of the functions f, f_1 and f_2.

q_1 and q_2 are the weights of the two distributions. Since $q_1 + q_2 = 1$, f has **five free parameters**. Exclude q_2 and denote $\mathbf{p}' = (q_1, \mu_1, \mu_2, \sigma_1, \sigma_2)$.

Introduce the **error function**

$$
C(\mathbf{p}') = \sum_i \left[f(i; \mathbf{p}') - P(i)\right]^2 \tag{5}
$$

To approximate $P(i)$ with $f(i;\mathbf{p}')$ and find the optimal parameters, we need to **minimise** $C(\mathbf{p}')$. This means nonlinear minimisation with 5 variables. Any nonlinear minimisation algorithm can be used, for example:

- Newton's method
- Marquard-Levenberg algorithm

Assume the optimal model function $f(i;\hat{\mathbf{p}})$ **has been obtained** and

$$\hat{\mathbf{p}} = (\hat{q}_1, \hat{q}_2, \hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2)$$

are the optimal parameters.

The optimal threshold can be derived analytically by minimising the **probability of erroneous classification**

$$E(t) = E_1(t) + E_2(t) = \int_{-\infty}^{t} f_2(i)di + \int_{t}^{\infty} f_1(i)di$$

Meaning of $E(t)$ for a candidate threshold t (see the above drawing):

- $E_1(t)$ is the probability that a pixel belonging to class 1 will be classified as belonging to class 2.
- $E_2(t)$ is the probability that a pixel belonging to class 2 will be classified as belonging to class 1.

Setting the derivative of $E(t)$ to zero and substituting f_1 and f_2 from (4), we obtain that the **optimal threshold** t_{opt} is a solution of

$$A \cdot t^2 + B \cdot t + C = 0, \quad (6)$$

where

$$\begin{aligned} A &= \hat{\sigma}_1^2 - \hat{\sigma}_2^2 \\ B &= 2(\hat{\mu}_1\hat{\sigma}_2^2 - \hat{\mu}_2\hat{\sigma}_1^2) \\ C &= \hat{\sigma}_1^2\hat{\mu}_2^2 - \hat{\sigma}_2^2\hat{\mu}_1^2 + 2\hat{\sigma}_1^2\hat{\sigma}_2^2 \ln\left(\frac{\hat{\sigma}_2\hat{q}_1}{\hat{\sigma}_1\hat{q}_2}\right) \end{aligned}$$

- **Two solutions** for t_{opt} are possible that minimise classification error.
- If the variances are equal, $\sigma_1^2 = \sigma_2^2 = \sigma^2$, a single optimal threshold exists:

$$t_{opt} = \frac{\hat{\mu}_1 + \hat{\mu}_2}{2} + \frac{\hat{\sigma}^2}{\hat{\mu}_1 - \hat{\mu}_2} \ln\left(\frac{\hat{q}_1}{\hat{q}_2}\right)$$

Algorithm 2: Gaussian threshold selection

1. Compute normalised histogram $P(i)$ of image $I(r,c)$.
2. Using a minimisation algorithm, minimise the error function $C(\mathbf{p}')$ defined by (5) and (4) and estimate the optimal parameters $\hat{q}_1, \hat{q}_2, \hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2$.
3. Solve equation (6) for t and obtain the optimal threshold t_{opt}.

Properties of the Gaussian mixture approach

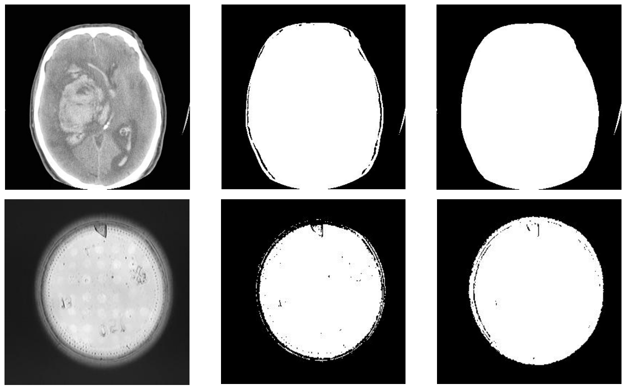
Advantages:

- Relatively general histogram model.
- When the model is valid, minimises classification error probability.
- Can be applicable small-size classes.

Drawbacks:

- Many histograms are not Gaussian. In particular, intensities are **finite** and **non-negative**.
  - A peak that is close to an intenisity limit cannot be approximated by Gaussian.
- Extension to multithresholding requires significant simplification of the model.
- It is difficult to detect close and flat modes.

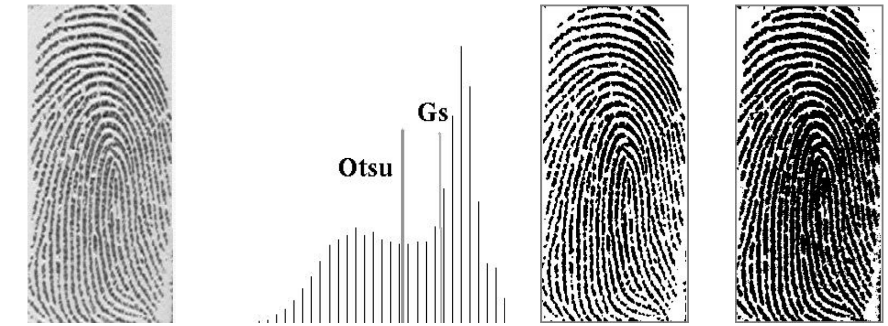
Examples of thresholding



images | Otsu results | Gaussian results

- Here, both methods give acceptable results.
- The Gaussian algorithm sets lower thresholds in both cases.

$\Rightarrow$ Fits object contours better than Otsu.



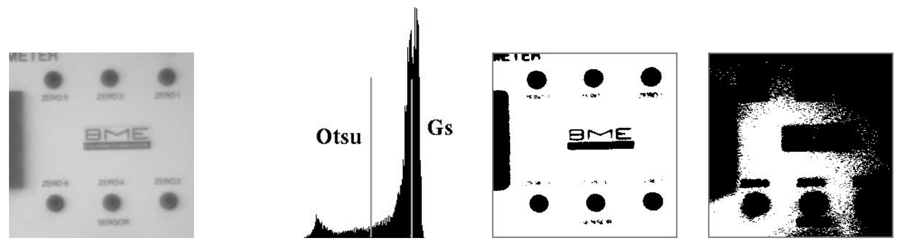
fingerprint image | image histogram | Otsu $T = 158$ | Gauss $T = 199$

- Here, both methods still give satisfactory results.
- The **Otsu** algorithm sets threshold in valley of histogram.

$\Rightarrow$ Fingerprint lines are well-separated.

- The **Gaussian** algorithm sets slightly high threshold.

$\Rightarrow$ Some fingerprint lines touch.



low-quality image | image histogram | Otsu $T = 159$ | Gauss $T = 201$

- Here, only Otsu gives a satisfactory result.
- The **Otsu algorithm** finds the small class of pixels (dark discs).
- The **Gaussian algorithm** tries to separate two high peaks formed by the background. Noisy valley is selected because the true class is
  - too small
  - too far

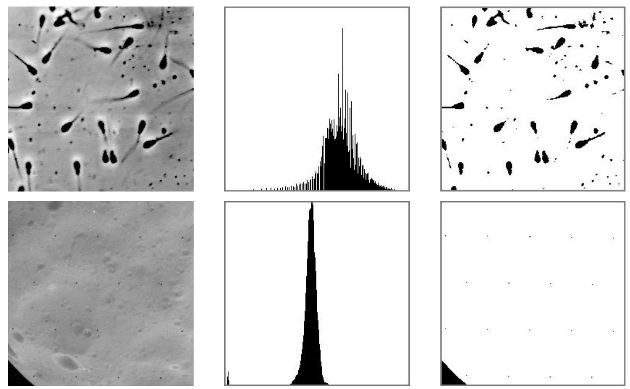


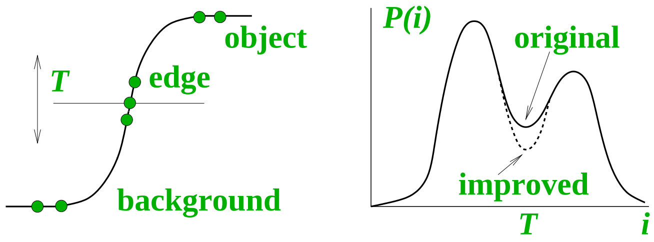
image | histogram | Otsu result

- Here, only Otsu algorithm **produces results**.
- Gaussian algorithm **gives no results** at all.
  - Upper row: unimodal histogram, no approximation obtained
  - Lower row: an approximation obtained, but the threshold equation has no real root.

Imroving the histogram for better peak separation

Use of gradient to improve histogram: Combine intensity and gradient information for better separation of objects and background.

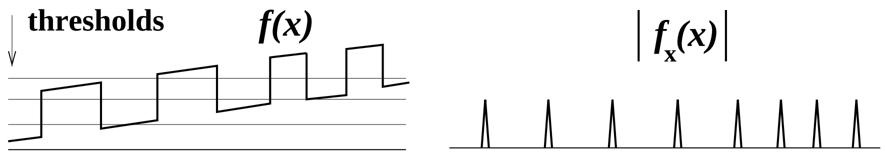
- Pixels close to edges have high gradients and medium intensities.
- Pixels of object and background have low gradient and low or high intensities.
- To better separate objects from background, **discard high gradient pixels** when computing the histogram.



Principle of histogram peak separation.

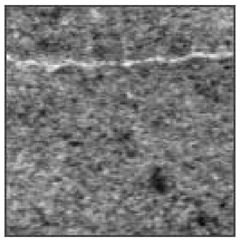
Thresholding versus edge detection

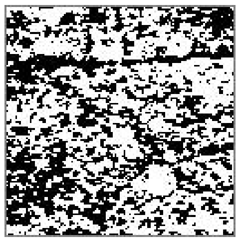
- **Thresholding** with a constant threshold is a global operation.
  - Advantage: Closed contours guaranteed
  - Drawback: Not applicable to images with uneven illumination
- **Edge detection** is a local operation
  - Advantage: Applicable to images with uneven illumination
  - Drawback: Closed contours not guaranteed

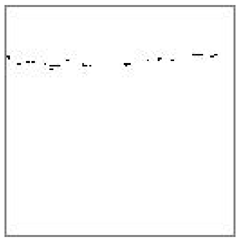


Signal with varying level that cannot be thresholded. Edges can be detected.

Limits of thresholding

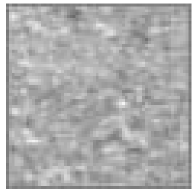

stone crack

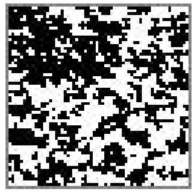

Otsu method

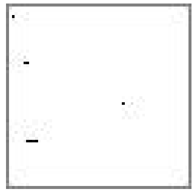

Another method

Merit (quality) of thresholding is **task-dependent**.

- The merit may include **geometric properties**.
- Image histogram does not account for geometry.
  - The crack is detected as set of bright pixels **independently of the crack shape**.


stone


Otsu method


Another method

In this example, the merit of thresholding is uncertain.

Limits of thresholding:

- No geometric informaton is taken into account.

⇒ Compact and connected regions are not guaranteed.
  - Select threshold, then arbitrarily interchange pixels in image, select again ⇒ **same threshold**

- Solution: Combine intensity and geometry using **region-oriented** methods.